Black hole formation from a complete regular past

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February 3, 2008

Abstract

An open problem in general relativity has been to construct an asymptotically flat solution to a reasonable Einstein-matter system containing a black hole in the future and yet past-causally geodesically complete, in particular, containing no white holes. We give such an example in this paper–in fact a family of such examples, stable in a suitable sense–for the case of a self-gravitating scalar field.

1 Introduction

The problem of gravitational collapse is typically formulated as the study of the future maximal evolution of complete asymptotically flat Cauchy data for an appropriate Einstein-matter system. The question of identifying physically admissible initial data, however, is best characterized by properties of their *past* evolution. In particular, it seems reasonable to restrict to Cauchy data whose past evolution is regular. Unfortunately, however, with the exception of certain classical results for dust [18], current theorems on the evolution of asymptotically flat Cauchy data ensuring a regular past [10, 9, 21] also ensure a regular future; this is due to the fact that these theorems depend on smallness in function-space norms that do not distinguish past from future.[1] As a consequence, even a single example of a solution to a reasonable Einstein-matter system with a regular asymptotically flat past and a singular future has thus far been lacking.

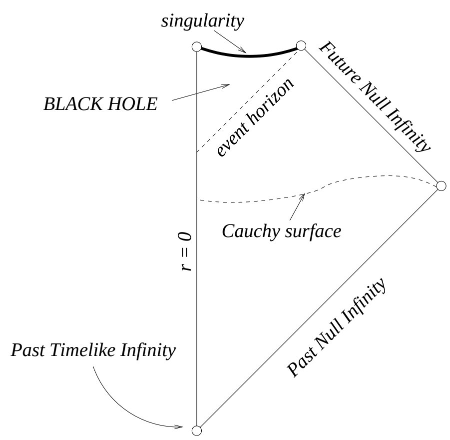
In the present paper, we shall prove

Theorem 1 *There exist past-causally geodesically complete solutions of the coupled Einstein-scalar field equations, which to the future terminate in a C^0-singularity hidden inside a black hole. These solutions are maximal developments of complete asymptotically flat Cauchy data.*

*supported in part by the NSF grant DMS-0302748

[1] One can compare with the cosmological case, where choice of gauge based on the expansion of the universe can be used to prove future completeness theorems in cases where the past is known to be singular. See for instance [1, 20].

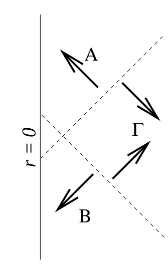
In fact, a large class of such solutions will be proven to exist. The solutions are spherically symmetric and their conformal diagram[2] is as follows:



Moreover, the above picture will be stable to a certain class of perturbations.

Although the solutions we construct will admit complete asymptotically flat Cauchy surfaces, they will not be constructed *a priori* as developments of such data. Rather, the examples of this paper will be constructed by pasting together the solutions to three distinct characteristic initial value problems. In particular, the issue of formulating a criterion on Cauchy data ensuring a singular future yet regular past is sidesteped in this paper.

Replacing the Cauchy problem with several distinct characteristic initial value problems allows one to completely separate the problem of the future from the past. The future is determined by data prescribed on a future-outgoing cone (see A in the diagram below), whereas the past is determined by data on a past-outgoing cone (see B). To complete the spacetime, however, one has to construct (and understand) the solution "between" the two cones (see Γ):



Since in spherical symmetry, the equations reduce to a quasilinear system in 2-dimensions, the problem depicted by Γ is also a well-posed characteristic initial value problem. Conceptually, this is perhaps easiest understood by replacing

[2]This is defined to be the image of a conformal representation of the quotient manifold $Q = M/SO(3)$ in a bounded domain of 2-dimensional Minkowski space.

the two-dimensional metric on Q by its negative, thus interchanging the notions of space and time.[3]

The data for the initial value problem B can be taken to be trivial, but alternatively, from Christodoulou [9, 7], one can prescribe an arbitrary scalar field sufficiently small in an appropriate norm, and this will also guarantee a past complete development. On the other hand, Christodoulou has given in [8] a criterion on initial data ensuring that for the initial value problem A, a black hole forms in the future. The important feature, from the point of view of this paper, is that this criterion can be applied to data which are not trapped initially, and for which the mass aspect function μ and the scalar field multiplied by the area radius $r\phi$ can be arbitrarily small in the supremum norm.

Theorem 1 thus reduces to completely understanding the evolution of problem Γ. It should be noted that the global features of this problem are quite different from the more standard characteristic initial value problems A and B; in particular, there are no *a priori* global energy bounds. The main analytical result of this paper is that Γ leads to a "complete" wedge in an asymptotically flat spacetime, given sufficiently small initial data. The requisite smallness, however, is asymmetric in the two initial characteristics; on the future outgoing characteristic, only pointwise smallness of μ and $r\phi$ are required. In particular, the requirements for the data for Γ are compatible with the data necessary in A to produce a black hole.

For completeness, in the next section, the Einstein-scalar field equations will be presented under spherical symmetry. The initial value problem Γ is studied in Section 3. From this, the theorem of this paper is easily obtained in Section 4 along the considerations outlined above.

Acknowledgement: The importance of identifying asymptotically flat initial data whose future development is singular but whose past development is regular has been stressed by Sergio Dain and Alan Rendall. I thank them for several very useful discussions.

2 The Einstein-scalar field equations under spherical symmetry

For a detailed discussion of the role of the Einstein-scalar field system in the study of the problem of gravitational collapse, the reader may refer to [13]. In rationalized units, the equations take the form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu},$$

$$g^{\mu\nu}\phi_{;\mu\nu} = 0,$$

$$T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi^{;\alpha}\phi_{;\alpha},$$

[3]Estimates in the direction depicted by Γ have many applications in the study of 2-dimensional hyperbolic problems. See [12, 16, 11, 4].

where $g_{\mu\nu}$ is a Lorentzian metric defined on a 4-dimensional manifold M and ϕ is a scalar function (the scalar field).

Recall that spherical symmetry is the assumption that $SO(3)$ acts by isometry on the spacetime and preserves ϕ. Under this assumption, the equations reduce to a second order system for functions (r, g_{ij}, ϕ) defined on the space of group orbits Q of the $SO(3)$ action. Here r is the so-called *area radius* function, i.e. r evaluated at a point p of Q retrieves up to a constant the square root of the area of the group orbit corresponding to p in M. This group orbit is necessarily a spacelike sphere. The tensor g_{ij} is a $1+1$-dimensional Lorentzian metric on Q, which is induced from the metric on M. The functions r and g_{ij} together determine $g_{\mu\nu}$ as follows:

$$g_{\mu\nu} dx^\mu dx^\nu = g_{ij} dy^i dy^j + r^2 \gamma,$$

where γ is the standard metric on the unit 2-sphere. Since the scalar field is constant on group orbits, it descends to a function on Q, and will still be denoted by ϕ.

One can prescribe null coordinates u and v on Q so that its metric becomes $-\Omega^2 du dv$. These coordinates then represent the characteristic directions of the equations. To exploit the method of characteristics, it is more convenient to write the equations as a first order system. Moreover, Ω can be replaced as an unknown in the equations by the Hawking mass

$$m = \frac{r}{2}(1 - |\nabla r|^2).$$

Introducing the first order derivatives $\partial_u r = \nu$, $\partial_v r = \lambda$, $r\partial_u \phi = \zeta$, $r\partial_v \phi = \theta$, and also the so-called mass aspect function $\mu = \frac{2m}{r}$, we obtain the system

$$\partial_u r = \nu, \tag{1}$$

$$\partial_v r = \lambda, \tag{2}$$

$$\partial_u \lambda = \lambda \left(\frac{2\nu}{1-\mu} \frac{m}{r^2} \right), \tag{3}$$

$$\partial_v \nu = \nu \left(\frac{2\lambda}{1-\mu} \frac{m}{r^2} \right), \tag{4}$$

$$\partial_u m = \frac{1}{2}(1-\mu) \left(\frac{\zeta}{\nu} \right)^2 \nu, \tag{5}$$

$$\partial_v m = \frac{1}{2}(1-\mu) \left(\frac{\theta}{\lambda} \right)^2 \lambda, \tag{6}$$

$$\partial_u \theta = -\frac{\zeta\lambda}{r}, \tag{7}$$

$$\partial_v \zeta = -\frac{\theta\nu}{r}. \tag{8}$$

From the above we also derive

$$\partial_v(\phi\nu + \zeta) = \frac{\phi}{r^2}\frac{2\lambda\nu}{1-\mu}m. \tag{9}$$

$$\partial_u(\phi\lambda + \theta) = \frac{\phi}{r^2}\frac{2\lambda\nu}{1-\mu}m. \tag{10}$$

We shall see that the r^{-2} factor on the right hand side above allows us to show that the quantities $\phi\lambda + \zeta$, $\phi\lambda + \theta$ have better decay in r than ϕ, θ, or ζ; this fact plays an important role in our argument.

We also easily obtain from the above the equations

$$\partial_u \frac{\lambda}{1-\mu} = \frac{1}{r}\left(\frac{\zeta}{\nu}\right)^2 \nu \frac{\lambda}{1-\mu} \tag{11}$$

and

$$\partial_v \frac{\nu}{1-\mu} = \frac{1}{r}\left(\frac{\theta}{\lambda}\right)^2 \lambda \frac{\nu}{1-\mu}. \tag{12}$$

3 The characteristic initial value problem Γ

In this section, we will study a characteristic initial value problem posed in the direction indicated by Γ in the Introduction. This will be the main new analytic element in the proof of Theorem 1 in the next section.

For the time being, fix constants $R > 0$, $C > 0$, $M > 0$, and $m_0 \geq 0$. The initial characteristic segments will be $u = -R$ and $v = R$. At $(-R, R)$, we prescribe

$$r(-R, R) = R,$$

$$m(-R, R) = m_0.$$

On $u = -R$, we prescribe $\mathcal{C}^2$ functions of v $r(-R, v) : [R, \infty) \to \mathbf{R}$, $\phi(-R, v) : [R, \infty) \to \mathbf{R}$ so as to satisfy

$$\partial_v r = \lambda = 1, \tag{13}$$

$$|r\phi| \leq C, \tag{14}$$

$$m \leq M. \tag{15}$$

On $v = R$, we prescribe $\mathcal{C}^2$ functions $r(u, R) : (-\infty, -R] \to \mathbf{R}$, and $\phi(u, R) : (-\infty, -R] \to \mathbf{R}$, so as to satisfy

$$\partial_u r = \nu = -1, \tag{16}$$

$$|\partial_u(r\phi)| = |\zeta + \nu\phi| \leq C|u|^{-2}, \tag{17}$$

$$m \leq M. \tag{18}$$

We have the following

Proposition 1 *Consider the initial value problem described above. There exists a unique subset* $\mathbf{D} \subset (-\infty, -R] \times [R, \infty)$*, open in the subspace topology of the latter set, and a unique set of sufficiently regular functions* r, ϕ, *and* m *defined on* $\mathbf{D}$*, such that*

1. $-R \times [R, \infty) \cup (-\infty, -R] \times R \subset \mathbf{D}$ *and* r, ϕ, m *match with the data prescribed.*
2. *The functions* r, ϕ, m *satisfy (1)–(8).*
3. *If* $(u, v) \in \mathbf{D}$*, then* $(\tilde{u}, \tilde{v}) \in \mathbf{D}$*, for all* $\tilde{u} \geq u$, $\tilde{v} \leq v$.
4. *Given any other* $\tilde{\mathbf{D}}$*, and sufficiently regular* $\tilde{r}$, $\tilde{m}$, $\tilde{\phi}$*, satisfying properties 1, 2, and 3, it follows that* $\tilde{\mathbf{D}} \subset \mathbf{D}$*, and* $\tilde{r} = r$, $\tilde{m} = m$*, and* $\tilde{\phi} = \phi$ *on* $\tilde{\mathbf{D}}$.

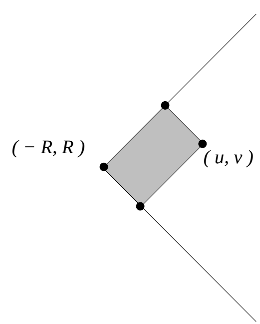
Moreover, if the initial data are C^∞*, then* r, ϕ, m *are* C^∞ *functions on* $\mathbf{D}$.

Proof. The proof is by standard techniques and is omitted. □

We shall call $\mathbf{D}$ the maximal development of our initial value problem. Time orienting the negative of the induced metric $-g_{ij}$ of our solution in the obvious way, property 2 above says that $\mathbf{D}$ is a past set with respect to $-g$. In what follows, we will denote causal relations with respect to the negative metric under the above time orientation by J^+_{-g}, J^-_{-g}, etc.

Proposition 2 *For the intial value problem described above,* $\nu < 0$ *and* $\lambda > 0$ *throughout* $\mathbf{D}$.

Proof. By continuity, the above inequalities indeed hold in a neighborhood of initial data. Thus, assuming the proposition is false, there exists a (u, v) such that either $\nu(u, v) = 0$ or $\lambda(u, v) = 0$, but for which $\nu < 0$, $\lambda < 0$ in $J^-_{-g}(u, v)$:



Let us first assume that $\lambda(u, v) = 0$. Integrating equation (11) on $[u, -R] \times \tilde{v}$ for each $\tilde{v} \in [R, v]$, we obtain that

$$
\begin{aligned}
\frac{\lambda}{1-\mu}(u, \tilde{v}) &= \frac{\lambda}{1-\mu}(-R, \tilde{v}) e^{\int_{-R}^{u} \frac{1}{r} \frac{\zeta^2}{\nu} d\tilde{u}} \\
&= \frac{\lambda}{1-\mu}(-R, \tilde{v}) e^{-\int_{u}^{-R} \frac{1}{r} \frac{\zeta^2}{\nu} d\tilde{u}} \\
&\geq \frac{\lambda}{1-\mu}(-R, \tilde{v}),
\end{aligned}
$$

since $\nu \leq 0$ in $\overline{J^-_{-g}(u,v)}$. It then follows from the above that we have $1-\mu \geq 0$ in $\overline{J^-_{-g}(u,v)}$.

Integrating now (3) on $[u,-R]\times v$, we obtain

$$\begin{aligned}\lambda(u,v) &= \lambda(-R,v)e^{\int_{-R}^{u}\frac{2\nu}{1-\mu}\frac{m}{r^2}d\tilde{u}} \\ &\geq \lambda(-R,v) \\ &\geq 1.\end{aligned}$$

But this contradicts the assumption $\lambda(u,v)=0$. In an entirely similar fashion, integrating (12), the assumption $\nu(u,v)=0$ leads to a contradiction. The proposition is thus proven. $\Box$

Proposition 3 *In* $\mathbf{D}$*,*

$$r \geq R, \tag{19}$$

$$m \geq m_0,$$

$$\nu \leq -1, \tag{20}$$

$$\lambda \geq 1, \tag{21}$$

$$0 < 1-\mu \leq 1. \tag{22}$$

Proof. The proof is immediate from the previous proposition and the signs in the equations (1)–(6). $\Box$

Let $\partial\mathbf{D}$ denote the boundary of $\mathbf{D}$ in the topology of $(-\infty,-R]\times[R,\infty)$. By monotonicity, r and m extend to continuous functions on $\mathbf{D}\cup\partial\mathbf{D}$ valued in the extended real numbers, and causality relations can still be applied. We have the following extension proposition:

Proposition 4 *If* $p \in \mathbf{D}\cup\partial\mathbf{D}$*,* $J^-_{-g}(p)\subset\mathbf{D}$*, and* $r(p)<\infty$*,* $m(p)<\infty$*, then* $p\in\mathbf{D}$*.*

Proof omitted. $\Box$

The achronal (with respect to $-g$) structure of the boundary $\partial\mathbf{D}$ immediately yields the following

Corollary 1 *If* $\partial\mathbf{D}\neq\emptyset$ *then there exists a* $p\in\partial\mathbf{D}$ *such that either* $r(p)=\infty$ *or* $m(p)=\infty$*.*

We now proceed to the main result of this section:

Proposition 5 *Fix* $\alpha > 1$*. Consider an initial value problem of the type described before, where*[4]

$$R > \max\left\{\left(\frac{4\alpha}{\alpha-1}\right)^2, \left(\frac{1+16\alpha^4M}{\alpha}\right)^2, \frac{8\alpha^2M}{\log\alpha}, \frac{6\alpha^2C^2}{\alpha-1}, \frac{4\alpha M}{1-\alpha}\right\}.$$

[4] We have here not attempted to give a scale-invariant condition.

Then

$$\mathbf{D} = (-\infty, -R] \times [R, \infty),$$

and the following estimates hold:

$$|r\phi| \le \alpha C, \tag{23}$$

$$|\nu\phi + \zeta| \le (1 + \alpha^4 M)C|u|^{-2}, \tag{24}$$

$$-1 \ge \nu \ge -\alpha \tag{25}$$

$$1 \le \lambda \le \alpha \tag{26}$$

$$1 - \mu \ge (2\alpha)^{-1} \tag{27}$$

Moreover, if in addition

$$|\lambda\phi + \theta|(R, v) \le \tilde{C}v^{-2}, \tag{28}$$

then

$$|\lambda\phi + \theta|(R, v) \le (\tilde{C} + \alpha^4 MC)v^{-2} \tag{29}$$

holds as well.

Proof. Consider the region $\mathcal{R}$ defined to be the set of all p such that the inequalities:

$$|r\phi| < 2\alpha C \tag{30}$$

$$|\nu\phi + \zeta| < 2\alpha C|u|^{-\frac{3}{2}} \tag{31}$$

$$\nu > -2\alpha \tag{32}$$

$$\lambda < 2\alpha \tag{33}$$

$$1 - \mu > (2\alpha)^{-1} \tag{34}$$

$$m < 2\alpha M \tag{35}$$

hold for all $q \in J^-_{-g}(p)$. With the exception of (31) and (34), the above inequalities hold on the intial data segments regardless of the value of R. Inequality (31) holds since $R > 1$, and (34) holds since

$$R > \frac{4M}{1 - \alpha^{-1}}.$$

By Proposition 1, $\mathcal{R}$ is non-empty, and contains a neighborhood of the initial segments; moreover, it is easily seen to be open in the topology of $\mathbf{D}$. We shall show first that in $\overline{\mathcal{R}}$, (30)–(35) hold with α replacing 2α.

Let (u, v) thus be in $\overline{\mathcal{R}}$. It follows by continuity that (30)–(35) hold at $(u, v) \cup J^-_{-g}(u, v)$, where, however, the $<$ sign is replaced by the $\le$ sign.

We estimate $r\phi(u, v)$ as follows. Integrating the equation

$$\partial_u(r\phi) = \nu\phi + \zeta,$$

we obtain from (14) and (31)

$$\begin{aligned}
|r\phi(u,v))| &\leq |r\phi(-R,v)| + \int_u^{-R} |\nu\phi + \zeta| d\tilde{u} \\
&\leq C + 4C\alpha R^{-\frac{1}{2}} \\
&= C(1 + 4R^{-\frac{1}{2}}\alpha) \\
&< \alpha C
\end{aligned}$$

since $R > \left[4\alpha(\alpha-1)^{-1}\right]^2$.

On the other hand, integrating (9) and applying (17), (30), (32), (34), (35), we estimate

$$\begin{aligned}
|\nu\phi + \zeta|(u,v) &\leq |\nu\phi + \zeta|(u,R) + \int_R^v \frac{|\phi|}{r^2} \frac{2\lambda|\nu|}{1-\mu} m d\tilde{v} \\
&\leq C|u|^{-2} + \int_R^{r(u,v)} \frac{|\phi|}{r^2} \frac{2|\nu|}{1-\mu} m dr \\
&\leq C|u|^{-2} + 32C\alpha^4 M \int_R^{r(u,v)} \frac{dr}{r^3} \\
&\leq C|u|^{-2} + 16C\alpha^4 M r(u,R)^{-2} \\
&= [C + 16C\alpha^4 M]|u|^{-2} \\
&\leq CR^{-\frac{1}{2}}[1 + 16\alpha^4 M]|u|^{-\frac{3}{2}} \\
&< \alpha C|u|^{-\frac{3}{2}},
\end{aligned}$$

where for the last inequality we use that $R > \alpha^{-2}(1 + 16\alpha^4 M)^2$.

To estimate $\nu(u,v)$, we integrate (4), applying (16), (32), (34), (35), to obtain

$$\begin{aligned}
\nu(u,v) &\geq \nu(u,R) e^{\int_R^v \frac{2}{r^2} \frac{\lambda|\nu|}{1-\mu} m d\tilde{v}} \\
&\geq -e^{8\alpha^2 M \int_R^{r(u,v)} \frac{dr}{r^2}} \\
&\geq -e^{8\alpha^2 M R^{-1}} \\
&> -\alpha
\end{aligned}$$

since $R > 8\alpha^2 M (\log\alpha)^{-1}$. Similarly, from (3) we estimate $\lambda(u,v)$, applying (13), (33), (34), (35):

$$\begin{aligned}
\lambda(u,v) &\leq \lambda(-R,v) e^{\int_u^{-R} \frac{2}{r^2} \frac{\lambda|\nu|}{1-\mu} m d\tilde{u}} \\
&\leq e^{8\alpha^2 M \int_{r(u,v)}^{-R} \frac{dr}{r^2}} \\
&\leq e^{8\alpha^2 M R^{-1}} \\
&< \alpha.
\end{aligned}$$

For m, from (5) we compute

$$m(u,v) = m(-R,v) + \int_{-R}^u \frac{1}{2} \frac{1-\mu}{\nu} \zeta^2 d\tilde{u}$$

$$
\begin{aligned}
&\leq \quad M + \int_{-R}^{u} \frac{1}{2}\frac{1-\mu}{\nu}\left[|\zeta + \nu\phi| + |\nu\phi|\right]^2 d\tilde{u} \\
&\leq \quad M + \int_{-R}^{u} \frac{1-\mu}{\nu}\left[(\zeta + \nu\phi)^2 + \nu^2\phi^2\right] d\tilde{u} \\
&\leq \quad M + 4\alpha^2 C^2 \int_{-R}^{u} |u|^{-3} du + 4\alpha^2 C^2 \int_{-R}^{r(u,v)} \frac{dr}{r^2} \\
&\leq \quad M + 2\alpha^2 C^2 R^{-2} + 2\alpha^2 C^2 R^{-1} \\
&< \quad \alpha M,
\end{aligned}
$$

where for the last inequality, we use $R > \max\left\{6\alpha^2 C^2(\alpha-1)^{-1}, 1\right\}$.

Finally from this, we obtain $1-\mu = 1 - \frac{2m}{r} \geq 1 - \frac{4\alpha M}{R} > \alpha^{-1}$ since $R > 4\alpha M(1-\alpha^{-1})^{-1}$.

Since

$$J^-_{-g}(\overline{\mathcal{R}}) \subset \overline{J^-_{-g}(\mathcal{R})} = \overline{\mathcal{R}}$$

we have just shown that for all $(u,v) \in \overline{\mathcal{R}}$, the inequalities (30)–(35) hold throughout $J^-_{-g}(u,v)$ with α replacing 2α. Thus $(u,v) \in \mathcal{R}$, so

$$\mathcal{R} = \overline{\mathcal{R}}.$$

Since $\mathbf{D}$ is connected, and $\mathcal{R}$ is open, it follows that

$$\mathcal{R} = \mathbf{D}.$$

Suppose now $(u,v) \in \partial\mathbf{D}$. By continuity, we have $m(u,v) \leq \alpha M < \infty$, and

$$
\begin{aligned}
r(u,v) \quad &\leq \quad R + \int_{-R}^{u} \nu(\tilde{u}, v) d\tilde{u} \\
&\leq \quad R + (-R-u)\alpha \\
&< \quad \infty.
\end{aligned}
$$

Thus, by the Corollary to Proposition 4, it follows that $\partial\mathbf{D} = \emptyset$, and thus

$$\mathbf{D} = \mathcal{R} = (-\infty, -R] \times [R, \infty).$$

To complete the proof of the proposition, it remains to show (24), and, under the additional assumption (28), (29). Integrating (9) and applying (17), (23), (25), (27), (29), we obtain

$$
\begin{aligned}
|\nu\phi + \zeta|(u,v) \quad &\leq \quad |\nu\phi + \zeta|(u,R) + \int_{R}^{v} \frac{|\phi|}{r^2}\frac{2\lambda|\nu|}{1-\mu} m d\tilde{v} \\
&\leq \quad C|u|^{-2} + \int_{R}^{r(u,v)} \frac{|\phi|}{r^2}\frac{2|\nu|}{1-\mu} m dr \\
&\leq \quad C|u|^{-2} + 2C\alpha^4 M \int_{R}^{r(u,v)} \frac{dr}{r^3} \\
&\leq \quad C|u|^{-2} + C\alpha^4 M r(u,R)^{-2} \\
&= \quad (1+\alpha^4 M) C|u|^{-2}.
\end{aligned}
$$

This gives (24). On the other hand, assuming (28) and integrating (10), we obtain

$$\begin{aligned}
|\lambda\phi + \theta|(u,v) &\leq |\lambda\phi + \theta|(-R,v) + \int_u^{-R} \frac{|\phi|}{r^2}\frac{2\lambda|\nu|}{1-\mu} m d\tilde{u} \\
&\leq \tilde{C}v^{-2} + \int_R^{r(u,v)} \frac{|\phi|}{r^2}\frac{2\lambda}{1-\mu} m dr \\
&\leq \tilde{C}v^{-2} + 2C\alpha^4 M \int_R^{r(u,v)} \frac{dr}{r^3} \\
&\leq \tilde{C}v^{-2} + C\alpha^4 M r(-R,v)^{-2} \\
&= (\tilde{C} + C\alpha^4 M)v^{-2}.
\end{aligned}$$

This completes the proof. □

4 Proof of the Theorem

In this section, we shall combine Proposition 5 with previous results of Christodoulou to give the proof of Theorem 1.

Fix α, C and M, and let R be as in Proposition 5. On the ray $v = R$, $u \leq 0$, prescribe

$$m(0,R) = 0,$$

$$r = -u,$$

and an arbitrary $\mathcal{C}^2$ function $\phi(u)$ satisfying

$$\int_{-\infty}^0 |\partial_u^2(u\phi)| du < \epsilon_*, \tag{36}$$

$$|\partial_u(u\phi)|(u,R) < C|u|^{-2}, \tag{37}$$

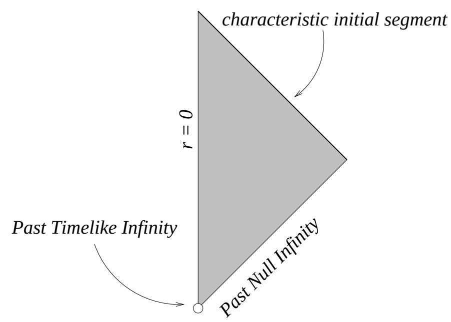
for $u \geq R$, and

$$|R\phi(-R,R)| < C,$$

$$\int_{-\infty}^0 \frac{1}{2}(\partial_u\phi)^2 u^2 < M. \tag{38}$$

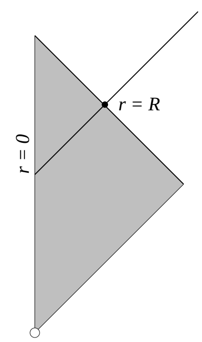
If $\epsilon_* > 0$ is sufficiently small, then, by Theorem 6.1 of [7], (36) implies that this initial value problem has a past-causally geodesically complete asymptotically

flat development:



When written as a first order system on Q, the boundary conditions imposed on the axis will be $r = 0$, $-u + v = R$. Moreover, if ϕ is $\mathcal{C}^\infty$, then this development is also $\mathcal{C}^\infty$.

From the above spacetime, consider the ray $u = -R$, and extend it off the shaded region to all positive values of v:



Where $v \geq R$, prescribe

$$r = v \tag{39}$$

and denote by $C_0 = |R\phi(-R, R)|$.

Now, choose some r_1 satisfying $R < r_1 < R+1$, and, setting $m(-R, R) = m_0$, choose some m_1 satisfying $m_0 < m_1 < M$. Define the function

$$I(V) = \min\left\{ \left(\frac{C - C_0}{2}\right)^2 \frac{1}{(V - r_1)^2}, \frac{M - m_1}{2R^2(V - r_1)} \right\}.$$

Fixing for the time being a $V > r_1$, from the definition of I, it is clear by a partition of unity argument that one can construct a large class of functions $\phi : -R \times [0, \infty) \to \mathbf{R}$, coinciding with the ϕ induced from the solution already obtained in $-R \times [0, R]$, and such that

1. The function ϕ, when viewed as a function of r on $[0, \infty)$, is regular, for instance $\mathcal{C}^\infty$.

2. The inequality

$$(\partial_v \phi)^2(u,v) > I(V), \tag{40}$$

holds for $v \in [r_1, V]$

3. The inequalities

$$|r\phi|(-R,v) < C, \tag{41}$$

$$m(-R,v) < M, \tag{42}$$

hold for all $v \geq R$, where m is defined by integrating (6).

We will denote the class of functions satisfying properties 1, 2, and 3 above as $\mathcal{F}_V$.

Define the quantities

$$\eta(V) = \min_{\phi \in \mathcal{F}_V} V^{-1}(m(-R,V) - m_1)$$

and

$$\delta(V) = V r_1^{-1} - 1.$$

It follows that

$$\begin{aligned}
\eta(V) &= \min_{\phi \in \mathcal{F}_V} V^{-1} \int_{r_1}^{V} \frac{1}{2}\theta^2 \frac{1-\mu}{\lambda} dv \\
&\geq \min_{\phi \in \mathcal{F}_V} R^2 V^{-1} \int_{r_1}^{V} \frac{1}{2}(\partial_v \phi)^2 dv \\
&\geq \frac{1}{2} R^2 V^{-1}(V - r_1) I(V) \\
&\geq \frac{1}{2} R^2 (R+1)^{-1}(V - r_1) I(V).
\end{aligned}$$

Thus for small enough $V - r_1$ we have

$$\eta(V) \geq \frac{M - m_1}{4(R+1)}. \tag{43}$$

Recall the function $E(y)$ defined in [8] by

$$E(y) = \frac{y}{(1+y)^2}\left[\log\left(\frac{1}{2y}\right) + 5 - y\right].$$

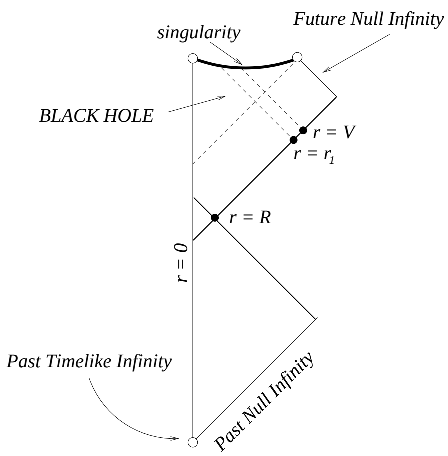
Note that $E(y) \to 0$ as $y \to 0$. Choose $V - r_1$ small enough so that

$$E(\delta) < \frac{M - m_1}{4(R+1)} \tag{44}$$

Inequalities (43) and (44) together imply

$$\eta > E(\delta). \tag{45}$$

From Theorem 5.1 of [8], property (45) ensures that initial data in $\mathcal{F}_V$ has an incomplete future development; moreover, its conformal structure is as depicted below:



(See [8] for details.) On the other hand, (37), (38), (41), and (42) imply that the rays $u = -R$, and $v = R$, and the functions r, ϕ, and m defined on them satisfy the conditions of Proposition 5. Applying this proposition one obtains a spacetime as depicted in the conformal diagram of the Introduction. It is easy to see that this spacetime admits an asymptotically flat Cauchy hypersurface. $\square$

The strict inequalities in (37), (38), (41), and (42) immediately indicate one notion according to which our solutions are stable in the spherically symmetric category. Alternatively, considering a Cauchy surface $\mathcal{S}$ intersecting $(-R, R)$, it follows that given any solution of the kind we have constructed, sufficiently small[5] perturbations supported on $\mathcal{S}$ in $r < R$ lead to a Cauchy development with a similar conformal diagram. For on the backwards characteristic, (36), (37), and (38) hold on account of Cauchy stability (this takes care of $r \leq R$) and the domain of dependence theorem (for $r \geq R$), thus guaranteeing the past completeness. On the other hand, Cauchy stability implies that $\eta > E(\delta)$ for these purturbations, as this refers to a compact set of the development. Thus, the formation of a black-hole in the future is also a stable feature, in this sense.

Another point is worth mentioning. As the formation of a black hole depends only on the solution on $-R \times [r_1, V]$, we can replace the part of the solution for $v \geq V$ with the solution of yet another characteristic initial value problem, where initial data are given on $v = V$ and on past null infinity for $v \geq V$. In particular, one can provide solutions with the conformal diagram indicated in the Introduction, for which there is no incoming radiation near spacelike infinity, i.e. for which m is constant along past null infinity in a neighborhood of spacelike infinity. On the other hand, by the results of [4], if past null infinity is complete and m is constant throughout, then either $m = 0$, in which case the solution is Minkowski space, or else it contains a white hole.

[5]For instance, in the $\mathcal{C}^\infty$ topology, if we consider only $\mathcal{C}^\infty$ solutions.

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